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USE OF METHODS OF SOLVING INVERSE PROBLEMS OF HEAT CONDUCTION TO ESTABLISH THE COEFFICIENT OF HEAT TRANSFER IN JET COOLING

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The problem of determining the coefficient of heat transfer is analyzed as an inverse problem for the heat-conduction equation. The results of a calculation of the coefficient of heat transfer on the basis of experimental data on the jet cooling of metal plates are presented.

The jet cooling of metal (cooling of a continuous ingot, thermal treatment of manufactured articles and intermediate products, etc.) is widespread in metallurgical production. Data on the local (effective) coefficient of heat transfer α as a function of the surface temperature T^{sur} of an ingot [1] are required for the calculation and design of systems and the attainment of the optimum modes of cooling. The appropriate data in the literature are inadequate, however.

The experimental determination of the boundary temperatures or fluxes needed for the reconstruction of α and the temperature field in the entire volume of a body is strongly hindered and sometimes impossible owing to the high intensity of heat exchange and the sharp fluctuations in the heat load characteristic of jet cooling. Moreover, it is often impossible to reliably measure the temperature of one or several internal points of the body. In such cases the problem of reconstructing the unknown boundary conditions is formalized in the form of an inverse problem of heat conduction, which has raised great interest in recent years (for example, see [2, 3] and the literature cited there).

We considered three approaches to the solution of this inverse problem, corresponding to the reconstruction of one of three kinds of boundary conditions at the heat-exchange surface. The relative effectiveness of the proposed methods and of the regular-regime method [4], which is widespread in engineering practice, is studied.

Within the framework of the one-dimensional model the process of heat propagation in a solidified ingot (plate) being cooled can be represented as a boundary-value problem for the heat-conduction equation

$$c\rho \frac{\partial T}{\partial \tau} = D \frac{\partial^2 T}{\partial x^2}, \quad 0 < x < L, \quad 0 < \tau \leqslant \theta, \tag{1}$$

$$T|_{\tau=0} = T^0(x), \quad 0 \leqslant x \leqslant L, \tag{2}$$

$$-D \frac{\partial T}{\partial x}\Big|_{x=L} = 0, \quad 0 < \tau \leqslant \theta \tag{3}$$

with one of the following boundary conditions at x = 0:

$$T|_{x=0} = T^{sur}(\tau), \quad 0 < \tau \leqslant \theta, \tag{4}$$

$$-D \frac{\partial T}{\partial x}\Big|_{x=0} = q(\tau), \quad 0 < \tau \leqslant \theta, \tag{5}$$

$$-D \frac{\partial T}{\partial x}\Big|_{x=0} = -\alpha(\tau)(T|_{x=0} - T^{w}(\tau)), \quad 0 < \tau \leqslant \theta. \tag{6}$$

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Here c, ρ, D, L, and θ are assigned constants; $T^0(x)$ and $T^w(\tau)$ are assigned functions.

The functions $T^{sur}(\tau)$, $q(\tau)$, and $\alpha(\tau)$ in (4)-(6) are not assigned a priori. Instead, the temperature $T^e(\tau)$ of the specimen at $x = l \in [0, L]$ is assumed to be assigned:

$$T|_{x=l} = T^e(\tau),\ 0 \leqslant \tau \leqslant \theta. \tag{7}$$

The problems of the reconstruction of the unknown functions $T^{sur}(\tau)$, $q(\tau)$, and $\alpha(\tau)$ from the conditions (1)-(7) belong to the class of inverse problems of heat conduction.

Many methods for minimizing the functionals estimating the discrepancy in the fulfillment of the condition (7) are effective algorithms for the solution of inverse problems. The square of the norm in L_2 of the departure of the calculated temperature $T|_{x=l}$ from the assigned "experimental" temperature $T^e(\tau)$ can serve as an example of such a functional:

$$J = \int_0^\theta [T|_{x=l} - T^e(\tau)]^2\, d\tau. \tag{8}$$

One of the functions $T^{sur}(\tau)$, $q(\tau)$, or $\alpha(\tau)$ serves as the argument of the functional (8), depending on whether the temperature $T(x, \tau)$ is the solution of the boundary-value problem (1)-(3) with a boundary condition of the first (4), second (5), or third (6) kind, respectively. In the last case the function $\alpha(\tau)$ realizing the minimum of the functional (8) is, under certain assumptions, the unknown coefficient of heat transfer. If the criterion (8) is minimized with respect to the quantities $T^{sur}(\tau)$ or $q(\tau)$, then the unknown coefficient of heat transfer $\alpha(\tau)$ is reconstructed from Eq. (6), where the temperature $T(x, \tau)$ is determined through the solution of the corresponding inverse problem of heat conduction. The dependence of the coefficient of heat transfer on the temperature of the surface being cooled can be found by eliminating the variable τ from the functions $\alpha(\tau)$ and $T(0, \tau) = T^{sur}(\tau)$ obtained.

Gradient methods of minimizing functionals are the most common. The use of these methods to minimize functionals of the type of (8) is complicated by the fact that the latter are not assigned explicitly but through the solution of some boundary-value problem. The technique developed in [5, 6] for calculating the gradients of such functionals allows one to find the value of the gradient using the so-called conjugate boundary-value problem, which is no more than twice as complicated as the determination of the value of the original functional in terms of the difficulty of the calculations. Making [5, 6] concrete, let us describe the algorithms for calculating the gradients of the analogs of the functional (8) for the three kinds of its enumerated agreements.

In the segments $0 \leq x \leq L$ and $0 \leq \tau \leq \theta$ we introduce grids (uniform, for simplicity of the presentation) with nodes $x_i = ih$, $i = 0, \ldots, N$; $\tau_j = j\delta$, $j = 0, \ldots, n$; $h = L/N$; $\delta = \theta/n$ (N and n are assigned natural numbers). The approximation of the value of $T(x_i, \tau_j)$ is designated as T_{ij}. The following equations are the difference analogs of Eqs. (1)-(3), constructed on an implicit four-point scheme with an approximation order $O(\tau + h^2)$ [7]:

$$c\rho\,\frac{T_{ij} - T_{ij-1}}{\delta} = D\,\frac{T_{i-1j} - 2T_{ij} + T_{i+1j}}{h^2},\ i = 1, \ldots, N-1, \tag{9}$$

$$j = 1, \ldots, n,$$

$$T_{i0} = T^0(x_i),\ i = 0, \ldots, N, \tag{10}$$

$$-D\,\frac{T_{Nj} - T_{N-1j}}{h} = \frac{h}{2}\,c\rho\,\frac{T_{Nj} - T_{Nj-1}}{\delta},\ j = 1, \ldots, n. \tag{11}$$

The difference analogs of the conditions (4)-(6) are the corresponding expressions

$$T_{0j} = T_j^{sur},\ j = 1, \ldots, n, \tag{12}$$

$$-D\,\frac{T_{1j} - T_{0j}}{h} = q_j + \frac{h}{2}\,c\rho\,\frac{T_{0j} - T_{0j-1}}{\delta},\ j = 1, \ldots, n, \tag{13}$$

$$-D\,\frac{T_{1j} - T_{0j}}{h} = -\alpha_j(T_{0j} - T_j^w) + \frac{h}{2}\,c\rho\,\frac{T_{0j} - T_{0j-1}}{\delta},\ j = 1, \ldots, n. \tag{14}$$

Here T_j^{sur}, q_j, α_j, T_j^w are the grid analogs of the functions $T^{sur}(\tau)$, $q(\tau)$, $\alpha(\tau)$, and $T^w(\tau)$, respectively. Equations (9)-(11), together with one of the conditions (12)-(14), form a

difference boundary-value problem from which, with assigned grid functions T_j^{sur}, q_j, or α_j, one can determine the grid function T_{ij}, by the trial-run method, for example.

The following implicitly assigned function of the variables T_j^{sur}, q_j, or α_j is the difference analog of the criterion (8):

$$J = \sum_{j=1}^{n} [T_{kj} - T^e(\tau_j)]^2 \delta, \tag{15}$$

where the natural number, k, $k \geq 1$, is chosen so that the grid node x_k approximates the point l while the grid function T_{ij} is the solution of the corresponding difference boundary-value problem. Everywhere below the gradient of the functional (15) is understood in the sense of the usual norm of the space of grid functions of the type of L_2 [6].

According to [6], the components of the gradient of the functional (15), treated as a function of q_j, $j = 1, \ldots, n$, can be found from the equations

$$\frac{\partial J}{\partial q_j} = \frac{2}{h} \lambda_{0j}, \quad j = 1, \ldots, n,$$

where λ_{ij} is the solution of the following conjugate system:

$$\begin{gathered}
-c\rho \frac{\lambda_{0j+1} - \lambda_{0j}}{\delta} = D \frac{2\lambda_{0j} + \lambda_{1j}}{h^2}, \quad j = 1, \ldots, n, \\
-c\rho \frac{\lambda_{1j+1} - \lambda_{1j}}{\delta} = D \frac{\lambda_{2j} - 2\lambda_{1j} - 2\lambda_{0j}}{h^2} - \\
- 2\delta (T_{kj} - T_j^e) \delta_{1k}, \quad j = 1, \ldots, n, \\
-c\rho \frac{\lambda_{ij+1} - \lambda_{ij}}{\delta} = D \frac{\lambda_{i-1j} - 2\lambda_{ij} + \lambda_{i+1j}}{h^2} - 2\delta (T_{kj} - T_j^e) \delta_{ik}, \\
i = 2, 3, \ldots, N-2; \quad j = 1, \ldots, n, \\
-c\rho \frac{\lambda_{N-1, j+1} - \lambda_{N-1j}}{\delta} = D \frac{\lambda_{N-2j} - 2\lambda_{N-1j} + 2\lambda_{Nj}}{h^2} - 2\delta (T_{kj} - T_j^e) \delta_{N-1,k}, \quad j = 1, \ldots, n, \\
-c\rho \frac{\lambda_{Nj+1} - \lambda_{Nj}}{\delta} = D \frac{\lambda_{N-1j} - 2\lambda_{Nj}}{h^2} - 2\delta (T_{kj} - T_j^e) \delta_{Nk}, \quad j = 1, \ldots, n, \\
\lambda_{in+1} = 0, \quad i = 0, 1, \ldots, N.
\end{gathered} \tag{16}$$

Here δ_{ij} is the Kronecker symbol.

If the grid function T_j^{sur}, $j = 1, \ldots, n$, is treated as the argument of the functional (15), then the components of the corresponding gradient are found from the equations

$$\frac{\partial J}{\partial T_j^{sur}} = -\lambda_{0j}, \quad j = 1, \ldots, n,$$

where λ_{ij} is the solution of the conjugate system (16), in which the equations for $i = 0$ and $i = 1$, $j = 1, \ldots, n$, have the following respective forms:

$$\lambda_{0j} = D \frac{\lambda_{1j}}{h^2}, \quad j = 1, \ldots, n,$$

$$-c\rho \frac{\lambda_{1j+1} - \lambda_{1j}}{\delta} = D \frac{\lambda_{2j} - 2\lambda_{1j}}{h^2} - 2\delta (T_{kj} - T_j^e) \cdot \delta_{1k}, \quad j = 1, \ldots, n.$$

If the argument of the functional (15) is the grid function α_j, $j = 1, \ldots, n$, then the components of the corresponding gradient are found from the equations

$$\frac{\partial J}{\partial \alpha_j} = -\frac{2}{h} \lambda_{0j} (T_{0j} - T_j^w), \quad j = 1, \ldots, n,$$

where λ_{ij} is the solution of the conjugate system (16) in which the equations for $i = 0$, $j = 1, \ldots, n$, are replaced by the following equations:

$$-c\rho \frac{\lambda_{0j+1} - \lambda_{0j}}{\delta} = D \frac{2\lambda_{0j} + \lambda_{1j}}{h^2} + \frac{2}{h} \lambda_{0j} \alpha_j, \quad j = 1, \ldots, n.$$

The ability to calculate the gradient of the functional (15) allows the use of various iteration methods of the gradient type for its minimization. In each iteration of the method one must solve two boundary-value problems for the heat-conduction equation: an initial and a conjugate problem. Numerical experiments on the solution of a number of model and applied inverse problems of heat conduction have shown the sufficiently high effectiveness of gradient methods (especially the method of steepest descent and Newton's method) in a wide range of variation of the initial approximations of the unknown mode. These experiments allow one to evaluate the role of the various parameters of the problem and the algorithm. In particular, the reconstruction of a boundary mode is accomplished faster and more accurately if the additional information is assigned closer to the corresponding boundary. The greatest error in the reconstruction of a boundary mode is observed at the initial and final times. The results are improved somewhat by the use of a gradient of the functional (15) defined in the sense of a norm differing from the norm in the space L_2 (for example, the norm in the space W_2^1) [8].

Experiments on model problems have shown the stability of these methods against random interference in the input data, with reliable agreement between the accuracy of the calculations and the accuracy of the input data and the error of the finite-difference approximation.

Agreement between the value of the functional (15) and the dispersion of the random interference within the assigned accuracy limits serves as an empirical criterion for stopping the iterations. This can be achieved through the appropriate regulation of the step of the iteration process (compare with [9], for example).

The proposed algorithms for the reconstruction of the boundary mode were used to analyze experimental data obtained in an investigation of the process of jet cooling of metal [10]. The experimental studies were conducted on a laboratory installation simulating the spray cooling of a continuous ingot.

Rectangular plates of brass and copper with a thickness L = 0.014 m were heated to a temperature of 750-800°C (uniform through the entire mass) in an electric furnace and were then cooled in a vertical position by the delivery of water from a pipe containing openings. The dimensions of the plate were such that the effect of edge cooling on the investigated temperature field was practically eliminated.

The readings of Chromel—Alumel thermocouples, whose junctions were caulked at several points along the length and at two levels through the thickness of the plates, were recorded with an N-700 oscillograph during the cooling process. The secondary instrument and the thermocouples were calibrated by measuring the temperatures of melts of aluminum and tin during their crystallization. The error in reading the oscillograms did not exceed ±5-10°.

The adequacy of the one-dimensional model lying at the basis of the calculation was estimated by reconstructing the temperature field of a plate and comparing the results of the calculation with experiment. A comparison of the experimental and calculated data on the temperature at one point (at $x = l_2$), obtained using the readings of a thermocouple at a different (in depth) point (at $x = l_1$) as the input data, showed that the disagreement between the actual and calculated temperatures does not exceed 5% on the average.

The surface temperatures obtained in the solution of the inverse problem by the three different methods (using boundary conditions of kinds I, II, or III) differ from each other by 5% on the average and by no more than 10%. An analysis of the numerical results shows that the values of the coefficient of heat transfer obtained in the solution of the inverse problem by the second and third methods (in the reconstruction of q_j and α_j) differ by an average of no more than 2% for the brass plate and by no more than 5% for the copper plate.

In the solution of the inverse problem by the first method (with the reconstruction of T_j^{sur}) the values of α_j are close to those obtained by the other two methods only for high enough surface temperatures (380-650°). For lower values of T^{sur} this method of determining α_j is connected with greater instability of the computational process, and it cannot be recommended for practical work.

The method of the direct reconstruction of α_j (the third method) has the greatest stability.

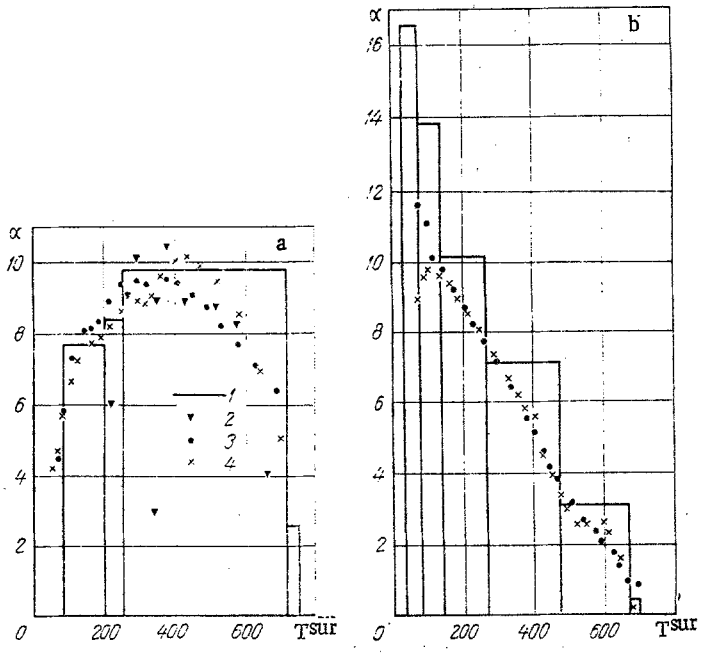

Fig. 1. Dependence of coefficient of heat transfer α (kW/cm^2•deg) on the surface temperature T^{sur} (°C): a) for copper plate; b) for brass plate: 1) from the results of a calculation by the steady-state method; from the results of a numerical solution of the inverse problem: 2) with an unknown boundary condition of kind I; 3) of kind II; 4) of kind III.

The dependence of the coefficient of heat transfer α on T^{sur} found by each of the three methods for the copper plate is shown in Fig. 1a, while this dependence found by the second and third methods for the brass plate is shown in Fig. 1b. Graphs of the functions $\alpha(T^{sur})$ obtained by the steady-state method for the copper and brass plates, respectively, are also presented.

It is seen from the figure that the steady-state method gives a satisfactory approximation of α in the cases under consideration.

We note that the method connected with the numerical solution of the inverse problem is applicable to a wide class of problems, particularly for linear problems with variable coefficients and for quasilinear problems of heat conduction.

For the fullest evaluation of the effectiveness of the methods under consideration in application to the process being studied we carried out a numerical solution of the direct problem of heat conduction (1)-(3) with the assignment of the boundary conditions (6) on the basis of the data of each method (Fig. 1). A comparison of the calculated and experimental data on the temperature $T(\zeta, \tau)$ showed that in the first case (with the assignment of α calculated by the steady-state method) the maximum disagreements between them were 13.3% for brass and 8.2% for copper, while for the methods connected with the numerical solution of the inverse problem they were 4.5 and 5.4%, respectively, in the range of T^{sur} = 90-720°C.

NOTATION

x, spatial coordinate, m; τ, time, sec; θ, final time; c, heat capacity; ρ, density; D, thermal conductivity; q, heat flux; T^w, temperature of cooling water; ζ, coordinate of point at which temperature is known; J, functional; h, spatial step of grid; δ, time step of grid; N, number of grid nodes in space; n, number of grid nodes in time; λ, solution of conjugate system.

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IMPURITY DISTRIBUTION IN A MELT CRYSTALLIZING WITH CONVECTION CAUSED BY CONCENTRATION GRADIENTS

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Numerical calculations on a model show that concentration-dependent convection has a marked effect on the impurity distribution in the solid.

Chemical nonuniformity arises in the crystallization of a melt because of partition of impurities in the two-phase medium when there is a moving phase interface; the exact distribution is also dependent on the mixing occurring in the core of a casting. Concentration-induced convection is one of the main causes of mixing.

Here it is assumed that the temperature differences arising at $\tau \neq 0$ on reducing the temperature of the boundaries to the crystallization point are insufficient to produce thermal convection in the melt.

The solubility difference between the solid and liquid phases causes spatial nonuniformity in the impurity pattern. The liquid core of the solidifying melt therefore shows convection whose direction is dependent on the density relationship between the impurity and the parent melt. We have made a numerical study of the impurity distribution occurring under such conditions.

The melt has an initial temperature T_0 (reasonably close to the crystallization point) and fills a rectangular semiinfinite region in which the dimensions of the vertical cross section are $L_1 \times L_2$ at time $\tau = 0$; at that instant the melt is immobile and the impurity and temperature are uniformly distributed over the cross section.

The impurity distribution in the initial solution is taken as being 0.1-0.3%, so we take the phase-transition boundary as being isothermal, while the crystallization front migrates into the liquid region in accordance with a square-root law. It is assumed that the solidification occurs in all directions at the same rate:

$$\varepsilon_1 = l_1 - \alpha\sqrt{\mathrm{Fo}},\ \varepsilon_2 = l_2 - \alpha\sqrt{\mathrm{Fo}},\ R_1 = R_2 = \alpha\sqrt{\mathrm{Fo}}.$$

The transport of momentum and of the impurity is described for an incompressible liquid in general by a system of equations that includes the Navier—Stokes equation, the equation for mass transport, and the equation of continuity.

The characteristic velocity and the characteristic pressure difference are defined by

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